Flavor Structure from Geometric Engineering

Dnyanesh Kulkarni
Cornell University

Based on work with M. Martone and Z. Yu

(To appear shortly 220X:XXXX)

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- Higher rank theories have so far resisted the challenge of classification. Yet, there is a developing understanding that the space of lower rank theories plays a critical role in exploring the moduli space structure at successively higher ranks and the moduli space techniques can shed light on various physical properties of the theories constructed via other methods.

[Argyres, Long, Martone, 2016; Martone, 2020; Argyres, Martone 2020]

Geometric Engineering

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- If the CY_3 is a hypersurface in either $\mathbb{C}^4$or $\mathbb{C}^3 \times \mathbb{C}^*$ with an isolated singularity, it is rather easy to extract quite a bit of physical information about the $\mathcal{N} = 2$ SCFT directly from the geometry (using the Jacobi algebra).

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- Given a simply laced Lie algebra J and a set of integers $\{b, k\}$, one can engineer two different types of theories: $J^b(k)$ and $D_k^b(J)$.

Geometric Engineering

- Notation:
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J	singularity	b
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D_N	$z_1^2 + z_2^{N-1} + z_2 z_3^2 + z_4^k = 0$	$2N-2$
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E_7	$z_1^2 + z_2^3 + z_2 z_3^3 + z_4^k = 0$	18
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- Here, J is part of the global symmetry of the corresponding 4d theory.
- Which other physical information does geometry provide us with?
- For a 4d theory $\mathcal{T}$, it is fairly simple to obtain in an algorithmic manner (using the Jacobi algebra of the IHS) the following: $(\mathcal{T}, c, r_{CB}, r_f, \{\Delta_{i,CB}\}_{i=1,..,r})$

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[Giacomelli, 2017]]

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- Which other physical information does geometry provide us with?
- For a 4d theory $\mathcal{T}$, it is fairly simple to obtain in an algorithmic manner (using the Jacobi algebra of the IHS) the following: $(\mathcal{T}, c, r_{CB}, r_f, \{\Delta_{i,CB}\}_{i=1,..,r})$
- However, J is only part of the full flavor symmetry algebra, and geometric engineering only provides us r_f.

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- This includes understanding the stratification of the Higgs branch, which for us is essentially a way of encoding the partial Higgsing pattern of the theory in a Hasse diagram. [Bourget, Cabrera, Grimminger, Hanany, Sperling, Zajac, Zhong, 2020]

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- For our purpose, it is sufficient to study partial Higgsing along the minilmal nilpotent orbit in order to determine the flavor symmetry. (This relies on the assumption that the Higgs branch chiral ring contains only flavor multiplets.) [Kaidi, Martone 2021]

Toolkit

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- Central charge formulae: generalization of Shapere-Tachikawa formulae, obtained by studying the $U(1)_r$ anomaly in a topologically twisted SCFT

  [Martone, 2020]

- Flavor level doubling rule

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- Anomaly matching relation for the c central charge of the theory obtained via partial Higgsing to that of the parent theory

  [Giacomelli, Meneghelli, Peelaers, 2020; Distler, Martone, to appear; Beem, Martone, Meneghelli, Peelears, Rastelli, to appear]

- Sugawara central charge condition for the associated 2d VOA

  [Beem, Peelaers, Rastelli, van Rees, 2016]

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$$24a = 5r + h + 6\left(\Sigma_{l=1}^{r}\Delta_l - r\right) + \Sigma_{i \epsilon I}\,\Delta_i^{sing}\,\frac{12c_i - h_i - 2}{\Delta_i}$$

$$12c = 2r + h + \Sigma_{i \epsilon I}\,\Delta_i^{sing}\,\frac{12c_i - h_i - 2}{\Delta_i}$$

This immediately gives the quaternionic dimension of the Higgs branch:

$$d_{\mathbb{H},HB} = 24\,(c - a)$$

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$$k_{\mathfrak{f}} = 2\,\Delta_{max}$$

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$$12\, c_{\mathcal{T}_{parent}} = 12\, c_{\mathcal{T}_{higgsed}} + 3k_{\mathfrak{f}} + \dim_{\mathbb{H}} \bar{\mathfrak{S}} - 3$$

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$$c_{2d,Sugawara} = \sum_i \frac{k_{2d} \dim G_i}{k_{2d} + h_i^\vee} \cong -12\, c_{4d}$$

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- We focus here on rank higher than 2, where our understanding of the moduli spaces is still developing.
- Using geometric engineering, one can immediately construct higher rank theories with the J, b, k as inputs and obtain aforementioned physical data.
  - A high fraction of these examples turn out to be Argyres – Douglas theories.

A rank-3 example: $D_2^5(A_5)$

Defining properties:

$$12c = 48$$

$$\Delta_i \in \left\{\frac{3}{2}, \frac{5}{2}, \frac{7}{2}\right\}$$

$$r_{\mathfrak{f}} = 6$$

$$J = A_5$$

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as well as $12c$ and $d_{\mathbb{H}}$ so as to satisfy the Anomaly matching central charge formula.

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It has $12c = 24$, $d_{\mathbb{H}} = 6, \mathfrak{f} = \mathfrak{su}(5)_5$.

Also, since $\dim(\mathfrak{su}(7)) = 48$ and $\mathrm{h}^\vee = 7$,
the Sugawara central charge matches our prediction of flavor symmetry and level:

$$c_{2d,Sugawara} = -48 = -12c_{4d}$$

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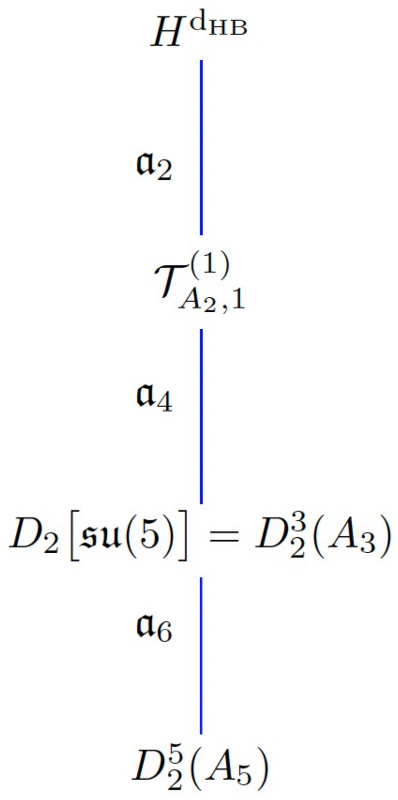
$$\mathfrak{f} = \mathfrak{su}(7)_7$$



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Defining properties:

$$12c = 27$$

$$\Delta_i \, \epsilon \, \left\{\frac{4}{3}, \frac{5}{3}, \frac{7}{3}\right\}$$

$$r_{\mathfrak{f}} = 3$$

$$J = A_2$$

$$24\, a = 50$$

$$d_{\mathbb{H},HB} = 4$$

$$k_{\mathfrak{f}} = \frac{14}{3}$$

$$\mathfrak{f} = \mathfrak{su}(4)_{\frac{14}{3}}$$

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$$\mathfrak{f} \neq \mathfrak{su}(4)_{\frac{14}{3}}$$

The Sugawara central charge doesn't match!!

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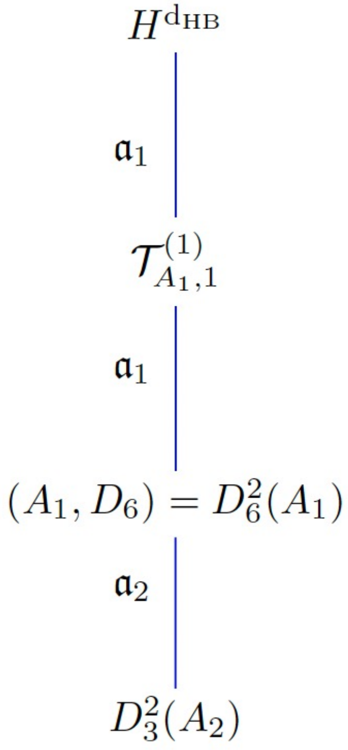

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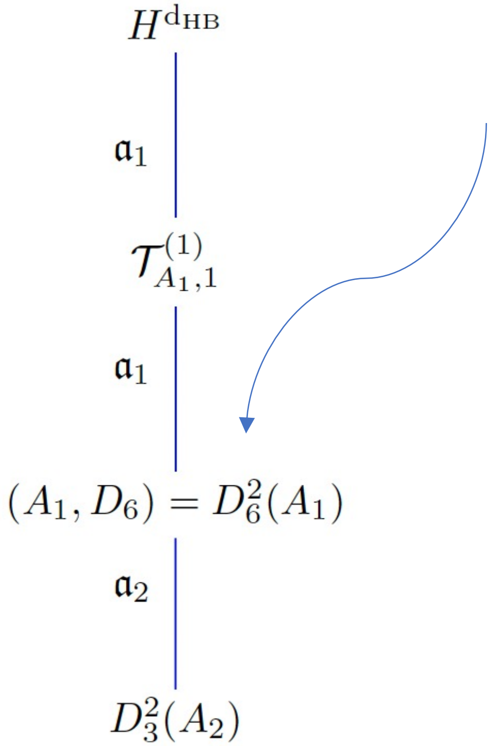

This theory has:

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- We conjecture and provide evidence that the VOA's associated to these theories are finite extensions of Affine Kac-Moody (AKM) algebras.
- We also check the consistency of our partial Higgsing results by recovering Schur indices of the daughter theories by applying an *index Higgsing procedure* to the parent indices. But that is a story for another time!

[Gaiotto, Rastelli, Razamat, 2013; Nishinaka, Sasa, Zhu, 2019; Beem, Peelaers, Rastelli, van Rees, 2016; Kaidi, Martone, 2021]

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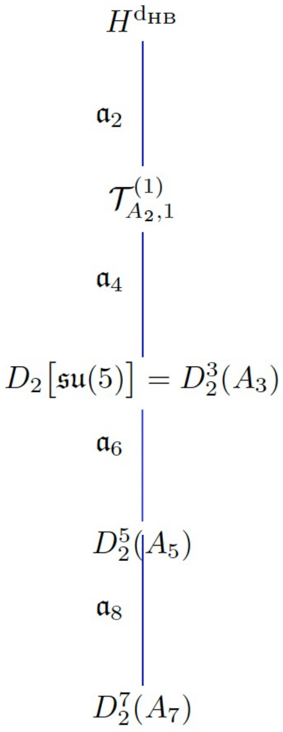

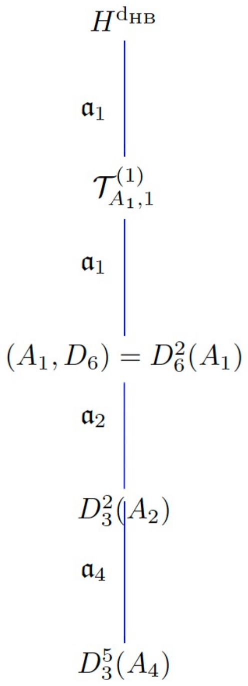

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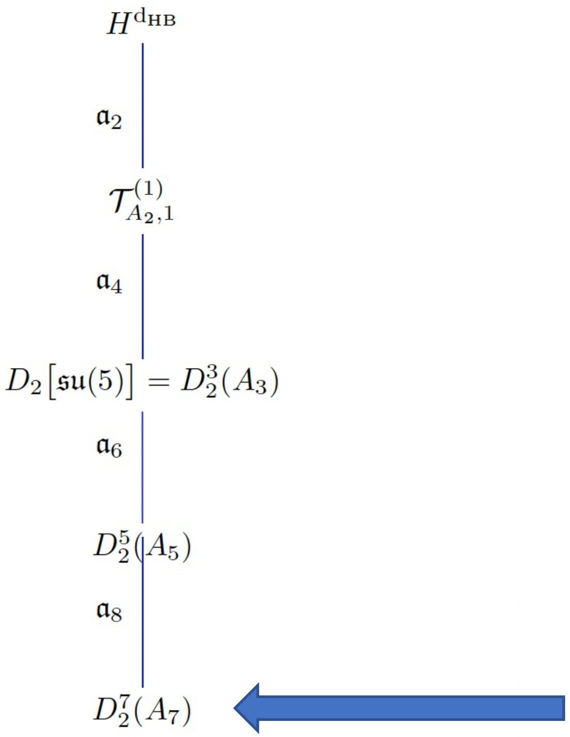

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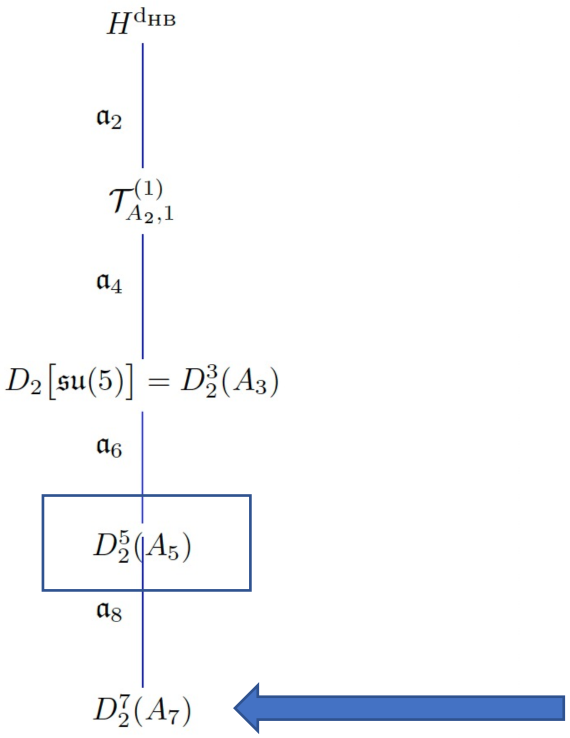

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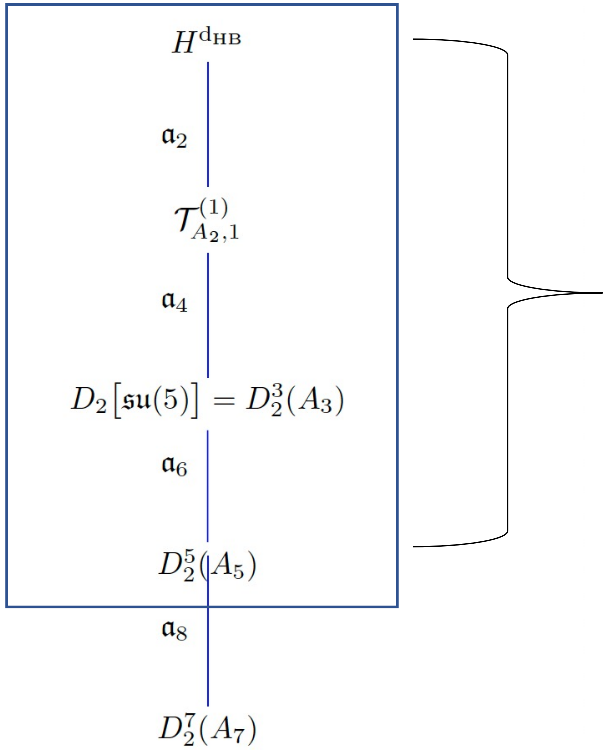

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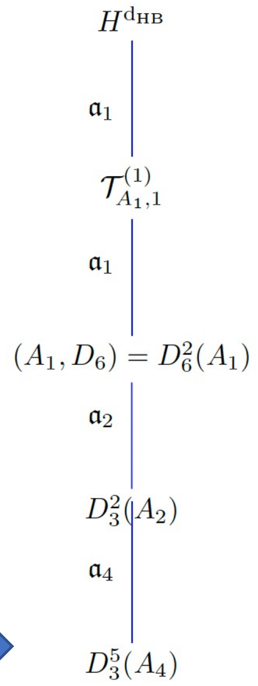

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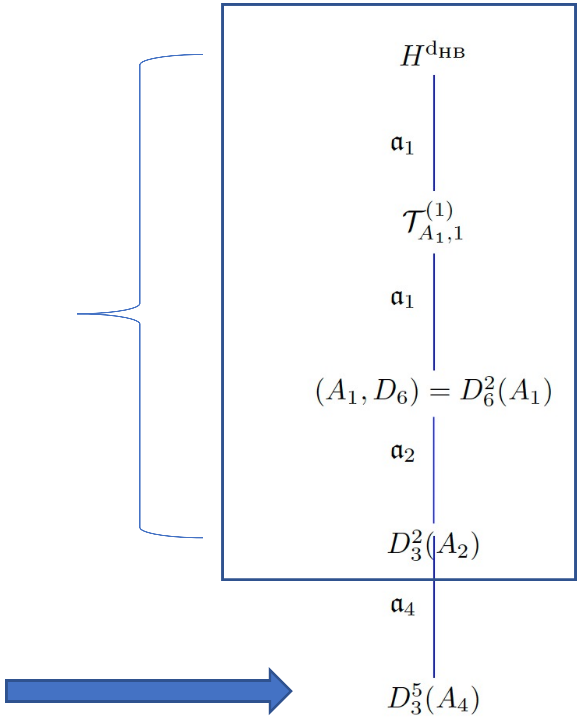

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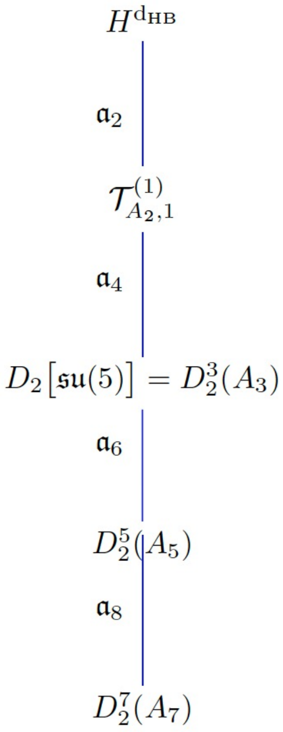

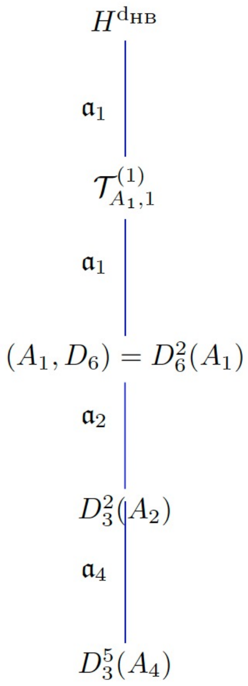

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- Coulomb branch stratification: can we get a handle on the singularity structure of the Coulomb branch of the class of theories presented today?

Thank you!